GEODESIC TRIANGULATIONS OF A STAR-SHAPED POLYGON

By

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Bachelor of Arts in Mathematics
Boston University
Boston, Massachusetts
2023

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
MASTER OF SCIENCE
December, 2025

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STAR-SHAPED POLYGON

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ACKNOWLEDGMENTS

I would like to thank my parents for their love and support. In addition, I would also like to thank my thesis advisor, Dr. Yanwen Luo, for his guidance.

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Date of Degree: DECEMBER, 2025

Title of Study: GEODESIC TRIANGULATIONS OF A STAR-SHAPED POLYGON

Major Field: MATHEMATICS

Abstract: Let $X(\Omega, T)$ denote the space of geodesic triangulations of Ω with combinatorial type T, where T is a triangulation of a convex polygon Ω embedded in $\mathbb{R}^2$, and let $W(\Omega, T)$ be the weight space of (Ω, T). Using a framework established by Luo which gives a relation between Tutte's Embedding Theorem and $X(\Omega, T)$, we prove several new results regarding the motion and limiting behavior of a geodesic triangulation $\tau \in X(\Omega, T)$, which is the image of a weight $w \in W(\Omega, T)$ under the Tutte map $\Psi : W(\Omega, T) \to X(\Omega, T)$, as we map τ to another geodesic triangulation of Ω by varying an interior-to-boundary edge weight within w.

Using these aforementioned results, we then give a possible approach toward studying the homotopy type of $X(\Omega, T)$, where Ω is strictly star-shaped with one reflex vertex and T is a triangulation of Ω which contains no dividing edges.

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CHAPTER I

INTRODUCTION

1.1 Introduction

One of the earliest results concerning the space of geodesic triangulations of a polygon embedded in the plane was established in the 1940s by Cairns [6], who proved that this space is path-connected if the polygon is a triangle. Several more results regarding the theory of geodesic triangulations were then proven in the following years, such as [2] and [10]; in particular, in the 1980s Bloch et al. [3] both strengthened and generalized Cairns' result above by showing that this space is contractible if the polygon is convex.

We are interested in studying the homotopy type of this space for a similar family of polygons, namely those which are strictly star-shaped. Luo showed in 2022 that this space is indeed contractible in the case of a strictly star-shaped quadrilateral [11]. However, one problem which currently remains open is the general case, specifically whether the space of geodesic triangulations of an arbitrary strictly star-shaped polygon with a triangulation containing no dividing edges is contractible.

The primary result of this thesis is as follows: Using a framework given in [12] – which connects geodesic triangulations to Tutte's Embedding Theorem – as our starting point, we construct a map which sends a geodesic triangulation τ of a convex polygon Ω to a similar geodesic triangulation of Ω, albeit with a new combinatorial type containing one less interior vertex. Moreover, we establish several new results regarding the behavior of τ under this mapping. Utilizing these results, we then give a possible approach toward settling the open problem above.

1.2 Preliminary Definitions

We begin by recalling several geometric definitions.

Definition 1.2.1 *Consider a map $\phi : V_B \to \mathbb{R}^2$, where $V_B = \{v_1, \ldots, v_{N_B}, v_{N_B+1} = v_1\}$, such that for every $1 \leq i \leq N_B$, each consecutive pair $\phi(v_i)$ and $\phi(v_{i+1})$ is connected by a line segment and the union of these segments form a simple closed curve in $\mathbb{R}^2$. The closure Ω of the component bounded by this simple closed curve is a **polygon** Ω **embedded in** $\mathbb{R}^2$, where V_B is the set of **boundary vertices of** Ω and the aforementioned line segments are its **boundary edges**.*

Henceforth, unless stated otherwise Ω will denote a polygon which is embedded in $\mathbb{R}^2$. Furthermore, note that Ω is a topologically closed disk.

Definition 1.2.2 *A **(simple) graph** is a tuple $G = (V, E)$ which consists of*

- *a set of **vertices** $V = \{v_1, \ldots, v_K\}$, and*
- *a set of **edges** E, where an edge is defined to be an unordered pair of distinct vertices $\{v_i, v_j\}$, with $1 \leq i, j \leq K$.*

From the above definition, a graph is a set-theoretic object, but we can regard it as a topological space constructed by gluing edges, identified as closed intervals in $\mathbb{R}$, along their vertices.

Definition 1.2.3 *A graph G is **planar** if it can be embedded in $\mathbb{R}^2$.*

Definition 1.2.4 *A planar graph G partitions $\mathbb{R}^2$ into path-connected open sets, called the **faces** f of G, with precisely one unbounded face.*

Definition 1.2.5 *A **triangulation of** Ω is a triple $T = (V, E, F)$ which consists of a set of vertices $V = V_B \sqcup V_I$, a set of edges E, and a set of faces F such that*

- *the one-skeleton $T^{(1)}$ of T, determined by V and E as a graph, is embedded in Ω and*

- *the boundary of each bounded face of T is a triangle.*

In terms of notation, we let the tuple (Ω, T) denote a polygon Ω with triangulation T. Moreover, in the above definition V_I denotes the set of **interior vertices** of (Ω, T) and $N_I := |V_I|$.

Definition 1.2.6 *Using the above notation, an edge in E is **strictly interior** if it connects two interior vertices and **interior-to-boundary** if it connects an interior vertex to a boundary vertex; in both cases, they are **interior** edges of (Ω, T).*

We now arrive at the most important definition in this thesis:

Definition 1.2.7 *Let (Ω, T) denote a triangulated polygon which is given by ϕ. A **geodesic triangulation of Ω with combinatorial type** T is an embedding $\tau : T^{(1)} \to \mathbb{R}^2$ of the 1-skeleton of T into $\mathbb{R}^2$ such that τ equals ϕ on V_B and maps every edge $e \in E$ to a line segment parametrized by unit speed. Let $X(\Omega, T)$ denote the space of geodesic triangulations of Ω with combinatorial type T.*

In particular, we say that τ in Definition 1.2.7 is a **straight-line embedding** of (Ω, T) which fixes the boundary polygon Ω.

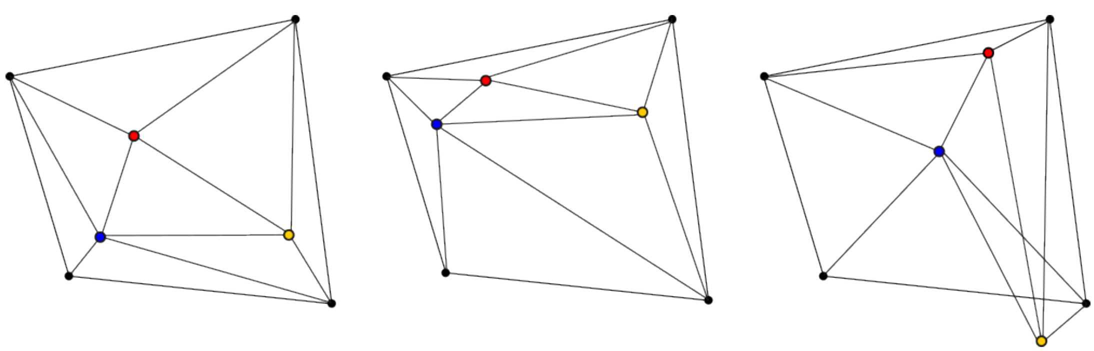

Figure 1: Only the two left-most diagrams are geodesic triangulations of a 4-gon Ω with a fixed combinatorial type T.

To gain a more formal understanding of $X(\Omega, T)$, we first remark that because every geodesic triangulation $\tau \in X(\Omega, T)$ is uniquely determined by the coordinates of its interior vertices in $\mathbb{R}^2$, then

Remark 1.2.1 *$X(\Omega, T)$ is a $2N_I$-dimensional manifold and is endowed with the Euclidean subspace topology from $\mathbb{R}^{2N_I}$.*

We now state a special family of triangulated polygons:

Definition 1.2.8 *Given a polygon Ω,*

- *An **eye** of Ω is a point $e \in \Omega$ such that for any other point $p \in \Omega$, the line segment connecting e to p is a subset of Ω;*
- *Ω is **strictly star-shaped** if it contains an eye, but not every point of Ω is an eye.*

Observe that if Ω is strictly star-shaped, then it contains a boundary vertex v_r with an interior angle greater than π. We refer to v_r as a **reflex vertex**.

Definition 1.2.9 *For an arbitrary triangulated polygon (Ω, T), a **dividing edge of** Ω is an interior edge of T which connects two boundary vertices.*

1.3 Historical Background

Regarding the study of geodesic triangulations for an arbitrary polygon Ω with triangulation T, one of the first results came in the 1940s when Cairns showed that

Theorem 1.3.1 (Cairns [6]) *For any triangulation T, $X(\Omega, T)$ is path-connected if Ω is a triangle.*

To that end, several years later Ho [10] improved Cairns' result above by showing that, with the same Ω as in Theorem 1.3.1, $X(\Omega, T)$ is simply-connected. Bloch et al. then strengthened this by proving

Theorem 1.3.2 (Bloch et al. [3]) *If Ω is convex, then $X(\Omega, T) \simeq \mathbb{R}^{2N_I}$.*

Corollary 1.3.1 *With the hypothesis in the preceding Theorem, $X(\Omega, T)$ is contractible.*

According to [3], one consequence of the above Corollary is the following classical result from topology due to Smale:

Theorem 1.3.3 (Smale [14]) *The space* Diff($\mathbb{D}^2$) *of diffeomorphisms of the 2-disk which fixes the boundary point-wise is contractible.*

Comparing an element of Diff($\mathbb{D}^2$) with those in Figure 2, we sometimes refer to the space of geodesic triangulations $X(\Omega, T)$ of a polygon Ω as a discretized approximation of Diff($\mathbb{D}^2$), the reason being that, informally speaking, as we keep triangulating Ω, thus increasing the number of interior vertices in the process, the embeddings will act on a larger and larger number of interior points, akin to the elements in Diff($\mathbb{D}^2$).

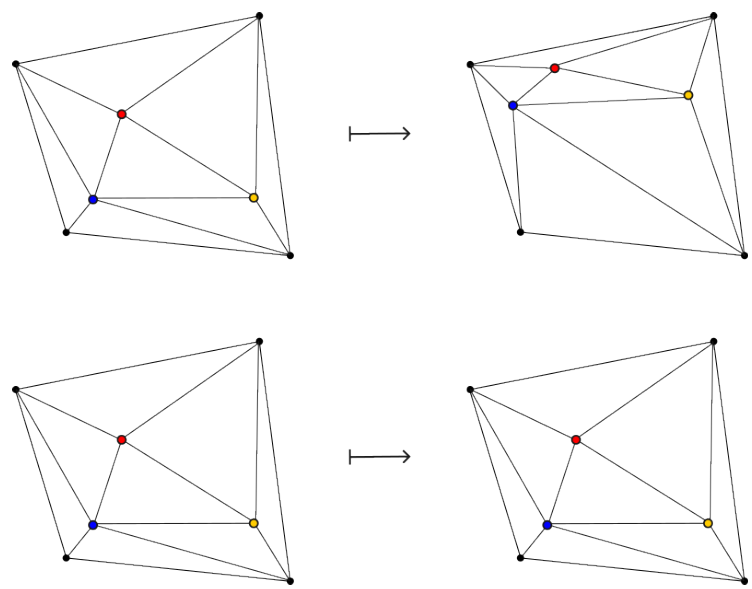

Figure 2: Two elements of $X(\Omega, T)$.

Restricting our attention to the case of star-shaped polygons, in 1978 Bing and Starbird proved that

Theorem 1.3.4 (Bing and Starbird [2]) *If* Ω *is strictly star-shaped and* T *has no dividing edges, then* $X(\Omega, T)$ *is both non-empty and path-connected.*

In fact, the example shown in Figure 3, which is a variant of the one constructed in [2], shows the importance of the strictly star-shaped hypothesis on Ω in order for the path-

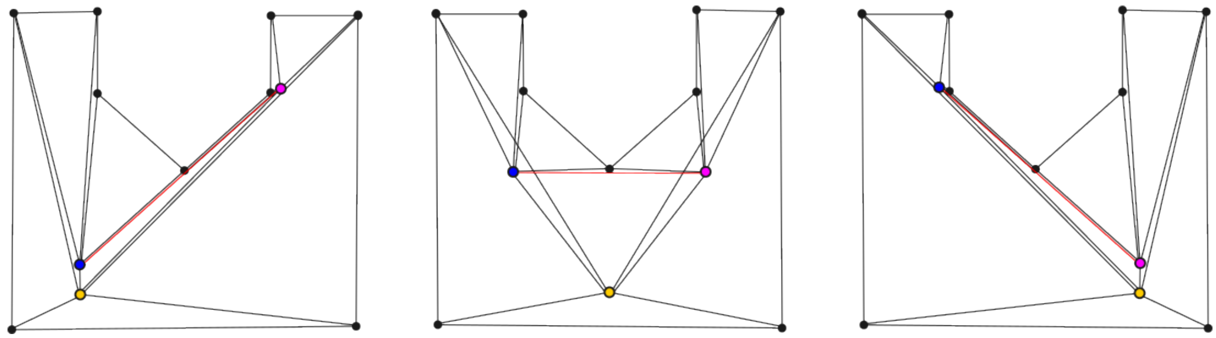

Figure 3: A triangulated non-star-shaped 9-gon (Ω, T) with $\pi_0 X(\Omega, T) \neq 0$.

connected conclusion to hold: In this case, Ω is a non-star-shaped nine-gon with a triangulation T specifying 3 interior vertices, v_1^I, v_2^I, and v_3^I, which are pictured in yellow, purple, and blue, respectively. The embeddings τ_1 and τ_2 of (Ω, T) on the left and right, respectively, are both elements of $X(\Omega, T)$. However, in order to isotope v_1^I in τ_1 to its position in τ_2, the interior edge e_{23}^{II} (which is depicted in red) connecting the two vertices v_2^I and v_3^I must eventually be horizontal as in the middle diagram to avoid colliding with an interior edge of v_1^I; but at the instant when this occurs, we obtain a degeneracy with some of the edges connected to v_1^I.

Furthermore, using non-star-shaped polygons, it was shown in 2022 that

Theorem 1.3.5 (Luo [12]) *For every $n > 0$, there exists a triangulated polygon (Ω, T) such that $\pi_n X(\Omega, T) \neq 0$.*

1.4 Problem Statement

In light of the results by Bing-Starbird and Luo above, the overarching conjecture of our thesis is as follows:

Conjecture 1.4.1 *If (Ω, T) is a triangulated star-shaped polygon with one reflex vertex and no dividing edges, then $X(\Omega, T)$ is contractible.*

1.5 Outline

In Chapter II, we recall a framework introduced by Luo in [12] which related an analogue of Tutte's Embedding Theorem to the study of geodesic triangulations in that it resulted in a new proof of Corollary 1.3.1; in particular, we review the definition of the weight space $W(\Omega, T)$ of (Ω, T) as well as the construction of the Tutte map $\Psi : W(\Omega, T) \to X(\Omega, T)$. Chapter III formalizes a proof sketch given in [11] which addresses a special case of Conjecture 1.4.1, namely when Ω is a strictly star-shaped quadrilateral. In Chapter IV, we then prove several new results pertaining to the behavior of any $\tau \in X(\Omega, T)$, where Ω is an arbitrary convex polygon with triangulation T, when we increase an interior-to-boundary edge weight within a weight $w \in W(\Omega, T)$ which is related to τ via $\tau = \Psi(w)$. Finally, in Chapter V we give a possible approach toward proving Conjecture 1.4.1 by applying several results from the previous Chapter.

CHAPTER II

TUTTE'S EMBEDDING THEOREM

Let (Ω, T) denote a convex polygon Ω with triangulation T. In [12], Luo gave a connection between Tutte's Embedding Theorem [15], which is a classical result from graph theory that gives a method for straight-line embedding certain planar graphs into $\mathbb{R}^2$ in a particular manner, to the space $X(\Omega, T)$. In particular, he used this link to give a new proof of Corollary 1.3.1, which states that $X(\Omega, T)$ is contractible.

In this chapter, we will review Tutte's theorem, an analogue of this theorem due to Floater [8], as well as the proof mentioned above, and several concepts from this Chapter will be used in Chapter IV.

2.1 Statement of Tutte's Theorem

Before stating this Theorem, we first review several definitions from graph theory.

Definition 2.1.1 *Given a graph $G = (V, E)$ and two distinct vertices $v_1, v_2 \in V$, a* ***path*** *from v_1 to v_2 is a sequence of edges which joins v_1 to v_2. If such a path exists, then v_1 is* ***connected*** *to v_2. In addition, G is* ***connected*** *if every pair of distinct vertices is connected.*

Definition 2.1.2 *A graph G is* ***k-vertex-connected*** *if, after the deletion of any set $A = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$ of fewer than k vertices in G, along with all of the edges (v_i, v_j) for which $v_i \in A$ or $v_j \in A$, the graph remains connected.*

Theorem 2.1.1 (Tutte [15]) *Given a 3-vertex-connected planar (simple) graph G such that its outer face is a convex polygon, there is a straight-line embedding of G into $\mathbb{R}^2$ such that each interior vertex of G is in the barycenter of its neighbors.*

In 2003, Floater [8] gave a reformulation of Theorem 2.1.1 in the case of a triangulated 2-disk, which is more relevant to our work as we are concerned with triangulated polygons. The following algorithm, which is referred to in [12] as **Tutte's Method**, is used to produce a geodesic triangulation of (Ω, T).

(1) Assign every interior directed edge e_{ij} of (Ω, T) a scalar $w_{ij} > 0$, which we refer to as the **edge weight** of e_{ij} (we need not require $w_{ij} = w_{ji}$);

(2) Fix the coordinates of each boundary vertex $v_i^B \in V_B$ of Ω;

(3) Solve the **Tutte system of equations** below, which consists of $2N_I$ equations, for the coordinates of the interior vertices $v_i^I \in V_I$:

$$\sum_{v_i^I \sim v_j} w_{ij}^I (v_i^I - v_j) = 0,$$

where $v_j \in \mathbb{R}^2$ ranges over all boundary and interior vertices of (Ω, T) and where $v_i^I \sim v_j$ means that v_i^I is adjacent to v_j. This solution exists and is unique;

(4) Place each interior vertex v_i^I in the plane according to the coordinates obtained in Step (3) and connect the vertices with line segments specifed by T. This yields an element of $X(\Omega, T)$.

2.2 Application to the Case of a Convex Polygon

In this section, we restate the proof of Corollary 1.3.1 given in [12] as well as the preceding concepts. As before, Ω will denote a convex polygon.

Definition 2.2.1 *An $N_I \times |V|$ matrix (w_{ij}) of edge weights w_{ij} is a* ***weight*** *of (Ω, T) if*

- $w_{ij} = 0$ *if* $v_i \nsim v_j$;
- $w_{ij} > 0$ *if* $v_i \sim v_j$.

Let $W(\Omega, T)$ denote the space of weights of (Ω, T), which we call its **weight space**.

Definition 2.2.2 *The* ***Tutte map*** *$\Psi : W(\Omega, T) \to X(\Omega, T)$ is defined by sending a weight (w_{ij}) to the (unique) geodesic triangulation specified by solving the Tutte system of equations with the edge weights of (w_{ij}) as coefficients.*

To give a topological grounding of the space defined above, we have that

Remark 2.2.1 *$W(\Omega, T)$ is a convex subset of the vector space $\mathcal{M}_{\mathbb{R}}(N_I \times |V|)$.*

The following map, which is due to Floater, gives a method for moving from the space of weights to the space of geodesic triangulations:

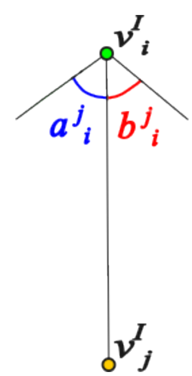

Figure 4: The Mean Value Coordinates.

Definition 2.2.3 (Floater [7]) *Let $\tau \in X(\Omega, T)$. For every directed interior edge e_{ij} of τ, define*

$$c_{ij} = \frac{\tan\left(a_i^j/2\right) + \tan\left(b_i^j/2\right)}{||v_i - v_j||},$$

where $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^2$ and a_i^j, b_i^j are depicted in Figure 4.

The ***Mean Value Coordinates*** *of each e_{ij} is defined as*

$$w_{ij} = \frac{c_{ij}}{\sum_{v_k^{\cdot} \sim v_i^I} c_{ik}}.$$

The following map gives us a section of Ψ:

Definition 2.2.4 *The **Floater map** $\sigma : X(\Omega, T) \to W(\Omega, T)$ sends a geodesic triangulation to a weight specified by the Mean Value Coordinates.*

One important observation regarding σ is that it is continuous.

Lemma 2.2.1 *The Tutte map Ψ is both continuous and surjective.*

Proof. **(Luo [12])** The function Ψ is continuous due to the continuous dependence of the solution of the Tutte system of equations on the coefficients of the system, whereas surjectivity follows from the fact that the Floater map σ is a section of Ψ. ■

We now state the proof of Corollary 1.3.1 given in [12].

Proof. **(Luo [12])** As σ constitutes a section of Ψ, by definition $\Psi \circ \sigma = \mathrm{Id}$. Because $W(\Omega, T)$ is convex, there is a homotopy

$$(1-t)(\sigma \circ \Psi) + t\mathrm{Id}$$

from $\sigma \circ \Psi$ to the identity map of $W(\Omega, T)$, so that $X(\Omega, T)$ and the contractible space $W(\Omega, T)$ have the same homotopy type. ■

CHAPTER III

STAR-SHAPED QUADRILATERALS

One result which addresses a special case of Conjecture 1.4.1 is due to Luo, who in [11] sketched a proof showing the contractibility of $X(\Omega, T)$, where Ω is a strictly star-shaped quadrilateral, by using concepts from projective geometry. The aim of this Chapter will be to formalize this aforementioned proof.

3.1 Projective Geometry

We begin by recalling an important theorem from the field of projective geometry, with the eventual goal of establishing a relation between two strictly star-shaped quadrilaterals which differ only by the location of their reflex vertices.

Theorem 3.1.1 *Given two quadrilaterals Q_1 and Q_2 which do not contain collinear vertices, there is a unique isomorphism $\phi : \mathbb{RP}^2 \to \mathbb{RP}^2$ taking Q_1 to Q_2 and which maps lines to lines.*

3.2 Proof of the Main Result

Theorem 3.2.1 (Luo [11]) *If Ω is a strictly star-shaped quadrilateral with reflex vertex v_r and T is any triangulation of Ω without dividing edges, then $X(\Omega, T)$ is contractible.*

Proof. Let $\widetilde{\Omega}$ denote the convex hull of Ω, which in this case is a triangle, and let $\widetilde{T}$ be the triangulation of $\widetilde{\Omega}$ induced by regarding the vertex v_r – which is now in the interior of $\widetilde{\Omega}$ – as an interior vertex of $(\widetilde{\Omega}, \widetilde{T})$, which we denote as $\tilde{v}_r$, and the two boundary edges of (Ω, T) connected to v_r as interior-to-boundary edges in $(\widetilde{\Omega}, \widetilde{T})$ (see Figure 5 below).

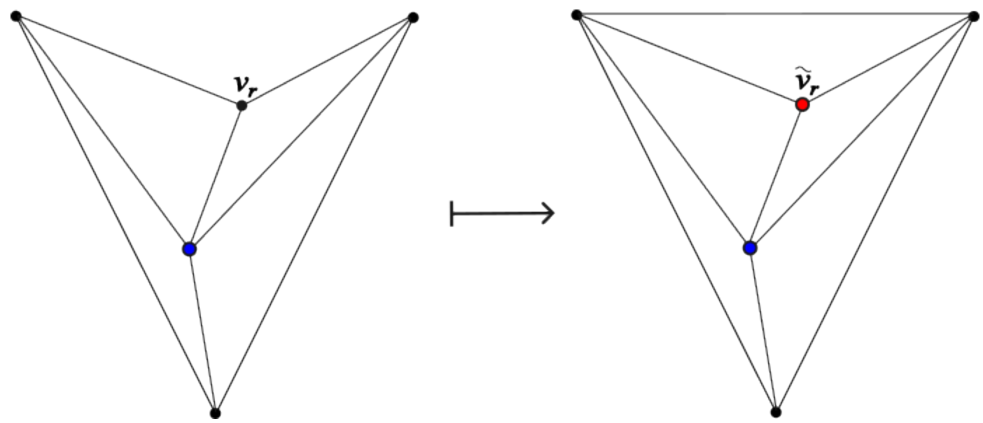

Figure 5: A geodesic triangulation of (Ω, T) induces a geodesic triangulation of $(\widetilde{\Omega}, \widetilde{T})$.

Let $\Delta = \text{Int}(\widetilde{\Omega})$ and define the projection map $\pi : X(\widetilde{\Omega}, \widetilde{T}) \to \Delta$ by sending a given $\tau \in X(\widetilde{\Omega}, \widetilde{T})$ to the coordinates of $\tilde{v}_r$ in τ, which we denote by $\tau(\tilde{v}_r)$. Observe that for any $v \in \Delta$, the preimage $\pi^{-1}(v)$ is precisely the subspace of $X(\widetilde{\Omega}, \widetilde{T})$ consisting of the $\tau \in X(\widetilde{\Omega}, \widetilde{T})$ for which $\tau(\tilde{v}_r) = v$; thus it is reasonable to identify this preimage with $X(Q_v, T)$, which is the space of geodesic triangulations of the quadrilateral Q_v which has the same boundary vertices as Ω except with v_r changed to v, meanwhile maintaining the original triangulation T.

Lemma 3.2.1 *Fix any Q_v as above and let ϕ_v be the unique isomorphism from Theorem 3.1.1 which sends Ω to Q_v. Then for any $\tau \in X(\Omega, T)$, the image $\phi_v(\tau) \in X(Q_v, T)$.*

Proof. To show that $\phi_v(\tau) \in X(Q_v, T)$, we want to prove that this graph is a straight-line embedding of (Q_v, T) which fixes Q_v. The map ϕ_v sends an interior edge e of τ to a line segment $\phi_v(e)$ by definition; as such, $\phi_v(\tau)$ is a straight-line embedding of $T^{(1)}$ in Q_v. Moreover, $\phi_v(e)$ is also an edge within the interior of $\phi_v(\Omega) = Q_v$ because, otherwise, it would intersect a boundary edge of Q_v, thus contradicting the one-to-one correspondence of ϕ_v. ■

One consequence of Lemma 3.2.1 is that the map $\phi_v : X(\Omega, T) \to X(Q_v, T)$ is a well-defined homeomorphism.

Corollary 3.2.1 *For every $v \in \Delta, X(\Omega, T) \simeq X(Q_v, T)$; in other words, $F := \pi^{-1}(v_r) \simeq \pi^{-1}(v)$.*

Lemma 3.2.2 *The structure $\big(\Delta, X\big(\widetilde{\Omega}, \widetilde{T}\big), \pi, F\big)$ is a trivial fiber bundle.*

Proof. Define the function $\Phi : X(\widetilde{\Omega}, \widetilde{T}) \to \Delta \times F$ by

$$\Phi(\tau) = (\tau(\tilde{v}_r), \widetilde{\tau}),$$

where $\widetilde{\tau} = \phi^{-1}_{\tau(\tilde{v}_r)}(\tau) \in X(\Omega, T)$. This function is continuous and is also injective because if $\tau_1(\tilde{v}_r) = \tau_2(\tilde{v}_r)$ and $\tilde{\tau}_1 = \tilde{\tau}_2$, then $\tau_1 = \tau_2$ as $\phi_{\tau_1(\tilde{v}_r)}$ is itself injective. The map Φ is also surjective because, given $(v, \tilde{\tau}) \in \Delta \times F$, there is an element $\tau \in \pi^{-1}(v)$ for which $\tilde{\tau} = \phi_v^{-1}(\tau)$ from surjectivity of ϕ_v. From this, we conclude that Φ is a global trivialization. ■

As a result, we obtain the product

$$X(\widetilde{\Omega}, \widetilde{T}) \simeq \Delta \times F. \tag{3.2.1}$$

Because $\widetilde{\Omega}$ is convex, then the left-hand of (3.2.1) must be contractible from Corollary 1.3.1. Since Δ is also contractible, we conclude that the same holds for $F = X(\Omega, T)$. ■

CHAPTER IV

GEODESIC TRIANGULATIONS

Let Ω be a convex polygon and T a triangulation of Ω. The goal of this chapter is to establish new results regarding how a geodesic triangulation $\tau = \Psi(w) \in X(\Omega, T)$ behaves, for some arbitrary $w \in W(\Omega, T)$, as we perturb a certain interior-to-boundary edge weight within w.

4.1 Varying the Weights

Consider a weight $w = (w_{ij}) \in W(\Omega, T)$ and fix any $v_n^B \in V_B$ with v_m^I an interior vertex adjacent to v_n^B. To reduce cumbersome notation in the following calculations, relabel $v_1^B := v_n^B$, $v_1^I := v_m^I$, and the corresponding edges and their edge weights.

Now for $t \geq 0$ variable, define a new class of weights $w(t)$ by

$$w_{ij}(t) = \begin{cases} w_{ij}^{IB} + t & \text{if } i = j = 1, \\ w_{ij} & \text{otherwise.} \end{cases}$$

Recall from Chapter 2 that there is a unique solution $\tau(t) := (v_1^I(t), \ldots, v_{N_I}^I(t)) \in X(\Omega, T)$, with $v_i^I(t) = (x_i^I(t), y_i^I(t))$, to the system

$$\sum_{v_j^I \sim v_i^I} w_{ij}^{II}(t)(v_i^I(t) - v_j^I(t)) + \sum_{v_j^B \sim v_i^I} w_{ij}^{IB}(t)(v_i^I(t) - v_j^B) = 0, \qquad (4.1.1)$$

for $i = 1, 2, \ldots, N_I$. As such, to formalize the goal at the beginning of this chapter we wish to examine the properties of $\tau(t)$ as $t \longrightarrow \infty$.

4.2 Matrix Representation of Tutte's System of Equations

The aim of this section is to study the aforementioned properties of $\tau(t)$ by taking advantage of several properties of certain families of matrices, such as that of an M-matrix and a weakly chained diagonally dominant matrix, whose definitions we recall below. To that end, our first goal will be to rewrite the system (4.1.1) above in matrix form in order to give an explicit equation for $v_i^I(t)$. Start by defining the $N_I \times N_B$ matrix of interior-to-boundary edge weights $W(t)$ by

$$W_{ij}(t) = \begin{cases} w_{ij}^{IB} + t & \text{if } i = j = 1, \\ w_{ij}^{IB} & \text{otherwise.} \end{cases}$$

Likewise, define the $N_I \times N_I$ matrix $S(t)$ by

$$S_{ij}(t) = \begin{cases} -\sum_{k \neq i}^{N_I} S_{ik} + \sum_{k=1}^{N_B} W_{ik}(t) & \text{if } i = j, \\ -w_{ij}^{II} & \text{if } v_i^I \sim v_j^I, \\ 0 & \text{if } v_i^I \not\sim v_j^I, \end{cases}$$

where $S_{ik} := S_{ik}(0)$. To shed some light on this matrix, observe that the sign of the diagonal entries S_{ii} are positive whereas for the off-diagonal entries it is non-positive. Moreover, S_{ii} is merely the sum of the edge weights for the edges connected to v_i^I: $S_{ii} = \sum_{k \sim i} w_{ik}(t)$.

With a computation similar to that in [11] – although we should note that the computation shown in [11] was only concerned with specific edge weights and did not involve a variable t – we can rewrite Equation (4.1.1) in matrix notation as

$$M(t)x(t) = b_x \text{ and } M(t)y(t) = b_y, \tag{4.2.1}$$

where $M(t)$ is the $(N_I + N_B) \times (N_I + N_B)$ block matrix

$$M(t) = \begin{bmatrix} S(t) & -W(t) \\ 0 & \text{Id} \end{bmatrix}$$

with $x(t) = [x_1^I(t) \ \cdots \ x_{N_I}^I(t) \ x_1^B \ \cdots \ x_{N_B}^B]^T$ and $b_x = [0 \ \cdots \ 0 \ x_1^B \ \cdots \ x_{N_B}^B]^T$ so that $y(t)$ and b_y are defined similarly.

At this point, we want to transition and briefly discuss a class of matrices with several useful properties that will allow us to eventually derive an explicit equation for $v_i^I(t)$.

Definition 4.2.1 *The* ***directed graph*** *of an $n \times n$ square matrix $A = (a_{ij})$ is a graph consisting of vertices $\{1, 2, \ldots, n\}$, where there is an edge connecting $1 \leq i, j \leq n$ if and only if $a_{ij} \neq 0$.*

Definition 4.2.2 *An $n \times n$ square matrix A is* ***weakly diagonally dominant*** *if for all $1 \leq i \leq n$,*

$$|A_{ii}| \geq \sum_{j \neq i}^{n} |A_{ij}|.$$

If we replace the inequality above with a strict inequality, we say that A is ***strictly diagonally dominant****.*

Definition 4.2.3 *A square matrix A is* ***weakly chained diagonally dominant*** *if*

1. *A is weakly diagonally dominant, and*

2. *For all rows i_1 that are not strictly diagonally dominant, there is a path from i_1 to i_k in the directed graph of A ending at a strictly diagonally dominant row i_k.*

Lemma 4.2.1 *For every $t \geq 0$, $S(t)$ is a weakly chained diagonally dominant matrix.*

Proof. For any $1 \leq i \leq N_I$, observe that

$$|S_{ii}(t)| = -\sum_{k \neq i}^{N_I} S_{ik} + \sum_{k=1}^{N_B} W_{ik}(t) \geq -\sum_{k \neq i}^{N_I} S_{ik},$$

so that Condition (1) of Definition 4.2.3 holds.

On the other hand, let i_1 be a row of the directed graph of $S(t)$ which is not strictly diagonally dominant; in other words, $\sum_{k=1}^{N_B} W_{i_1 k}(t)$ vanishes so that $v_{i_1}^I$ is not adjacent to any boundary vertex of (Ω, T). Because (Ω, T) is connected, there is a path from $v_{i_1}^I$ to some

$v^I_{i_k}$ such that this latter interior vertex is adjacent to a boundary vertex. Because row i_k is strictly diagonally dominant by definition, Condition (2) holds as well. ■

Because $S(t)$ is weakly chained diagonally dominant, then it is invertible by [13] and we can write the inverse of $M(t)$ as

$$M^{-1}(t) = \begin{bmatrix} S^{-1}(t) & S^{-1}(t)W(t) \\ 0 & \mathrm{Id} \end{bmatrix}.$$

As such, the solutions to (4.2.1) are

$$x^I(t) = S^{-1}(t)W(t)x^B \text{ and } y^I(t) = S^{-1}(t)W(t)y^B, \tag{4.2.2}$$

where $x^B = [x^B_1 \cdots x^B_{N_B}]^T$ and similarly for y^B. In other words, (4.2.2) expresses the solution $\tau(t)$ to (4.1.1) in terms of $S(t), W(t)$, and the boundary vertices.

4.3 Limiting Behavior of the Interior Vertices

We begin by stating two concepts which will relate to $S(t)$:

Definition 4.3.1 *A square matrix is an M**-matrix** if it has non-positive off-diagonal entries and eigenvalues with nonnegative real parts.*

Lemma 4.3.1 (Bramble [5]) *If A is a weakly chained diagonally dominant matrix with non-positive off-diagonal and positive diagonal entries, then it is an M-matrix.*

We use Lemma 4.3.1 to establish the following Proposition regarding the behavior of $\tau(t)$ as $t \to \infty$, namely we show that the interior vertices of $\tau(t)$ in fact converges.

Proposition 4.3.1 *The limit of $v^I_k(t)$ as $t \to \infty$ exists for every $1 \leq k \leq N_I$.*

Proof. For any $t \geq 0$, consider the column vectors $c = [1\ 0\ \cdots\ 0]^T$ and $b = t \cdot c$, both of which are of length N_I. Because $S := S(0)$ in particular satisfies the hypothesis of Lemma 4.3.1, then it is an M-matrix and thus by the monotonicity property of such a matrix we have that every entry in S^{-1} is non-negative. Consequently, $1 + c^T S^{-1} b = 1 + S_{11}^{-1} t > 0$. Because $S^{-1}(t)$ is the inverse of the sum of S and a rank-1 matrix, we utilize the Sherman-Morrison Formula [9] to obtain

$$S^{-1}(t) = (S + bc^T)^{-1} = S^{-1} - \frac{1}{1 + S_{11}^{-1} t} S^{-1} b c^T S^{-1}$$

$$= S^{-1} - \frac{t}{1 + S_{11}^{-1} t} S_{\cdot 1}^{-1} S_{1 \cdot}^{-1}.$$

Consequently, as $t \to \infty$ we have

$$S_{ij}^{-1}(t) = S_{ij}^{-1} - \frac{t}{1 + S_{11}^{-1} t} S_{i1}^{-1} S_{1j}^{-1} \longrightarrow S_{ij}^{-1} - \frac{S_{i1}^{-1} S_{1j}^{-1}}{S_{11}^{-1}}, \tag{4.3.1}$$

where $S_{11}^{-1} \neq 0$ by the following Lemma:

Lemma 4.3.2 $S_{11}^{-1} > 0$.

Proof. Multiplying the first row of S with the first column of S^{-1} yields

$$\left(\sum_{k \sim 1} w_{1k} \right) S_{11}^{-1} + S_{12} S_{21}^{-1} + \cdots + S_{1N_I} S_{N_I 1}^{-1} = 1.$$

Because $S^{-1} \geq 0$ and likewise for every $-S_{1i}$, then using the fact that at least one w_{1k} is non-zero – otherwise v_1^I would not be connected to any interior vertex – then solving for S_{11}^{-1} gives the desired inequality. ∎

Now consider the $N_I \times N_B$ intermediary matrix $A(t) = S^{-1}(t) W(t)$. Because

$$A_{ij}(t) = \sum_{k=1}^{N_I} S_{ik}^{-1}(t) W_{kj}(t),$$

and $W_{kj}(t)$ is constant except when $k = j = 1$, then in light of the convergence of Equation (4.3.1) we only need to check that $S_{i1}^{-1}(t) W_{11}(t)$ converges as well in order to complete our proof.

To that end, for any $1 \le i \le N_I$ a straightforward computation shows that

$$S_{i1}^{-1}(t)W_{11}(t) = \left(S_{i1}^{-1} - \frac{t}{1+S_{11}^{-1}t} S_{i1}^{-1} S_{11}^{-1}\right)(w_{11}^{IB} + t)$$

$$= S_{i1}^{-1}\left(w_{11}^{IB} + t - \frac{S_{11}^{-1}t(w_{11}^{IB}+t)}{1+S_{11}^{-1}t}\right)$$

$$= S_{i1}^{-1}\left(\frac{w_{11}^{IB}+t}{1+S_{11}^{-1}t}\right),$$

which limits to S_{i1}^{-1}/S_{11}^{-1} as $t \to \infty$, which again is well-defined from Lemma 4.3.2 above. As such, for every $1 \le i, j \le N_I$, $A_{ij}(t)$ converges as $t \to \infty$ and consequently the same holds for every $\lim_{t\to\infty} v_k^I(t)$. ■

As a direct result of the preceding Proposition, we obtain the following Corollary:

Corollary 4.3.1 *The interior vertex $v_1^I(t)$ converges to v_1^B as $t \to \infty$.*

Proof. Keeping the notation of the previous proof,

$$x_1^I(t) = \text{row}_1(A(t))x^B = \sum_{m=1}^{N_B}\left(\sum_{k=1}^{N_I} S_{1k}^{-1}(t)W_{km}(t)\right)x_m^B.$$

For $m = 1$, the inner sum limits to 1 as $t \to \infty$ since

$$A_{11}(t) = S_{11}^{-1}(t)W_{11}(t) + \sum_{k=2}^{N_I} S_{1k}^{-1}(t)W_{k1} \longrightarrow \frac{S_{11}^{-1}}{S_{11}^{-1}} + \sum_{k=2}^{N_I} 0 = 1,$$

and 0 for $m \ne 1$ because for all $1 \le k \le N_I$,

$$S_{1k}^{-1}(t)W_{km} \longrightarrow \left(S_{1k}^{-1} - \frac{S_{11}^{-1}S_{1k}^{-1}}{S_{11}^{-1}}\right)W_{km} = 0.$$

Repeating the same process for $y_1^I(t)$, we see that $(x_1^I(t), y_1^I(t)) \longrightarrow (x_1^B, y_1^B)$ as $t \to \infty$. ■

From Proposition 4.3.1, we showed that the interior vertices converges as $t \to \infty$. However, we made no mention of the behavior of the interior edges, which of course is salient in determining whether $\tau(t)$ is an embedding at the limit. In other words, we must consider the limiting behavior of the graph $\tau(t)$ in its entirety.

4.4 Limiting Behavior of the Interior Edges

Before considering this limiting behavior, observe that our choice of $v_m^I \in N(v_n^B) \cap V_I$, where $N(v)$ denotes the set of neighbors of the vertex v, in the previous section was arbitrary. However, by collapsing v_m^I we may inadvertently collapse a subcomplex of the subgraph of (Ω, T) which contains v_m^I and v_n^B as part of its boundary; see Figure 6 for such a subcomplex.

In the first part of this section, we prove that, in fact, there always exists a certain v_m^I for which this does not occur.

Definition 4.4.1 *A* ***triangle subgraph*** *is a subgraph of (Ω, T) which is the 1-skeleton of a triangulated triangle.*

One combinatorial remark we would like to make is the following: Let (Ω, T) be a triangulated convex polygon. A triangle subgraph of (Ω, T) which contains v_m^I and v_n^B as part of its boundary is the only subcomplex configuration S of (Ω, T) such that if we identify v_m^I with v_n^B, then S will be identified to a line segment.

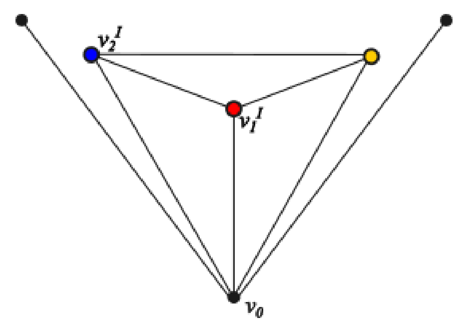

Figure 6: Taking $v_2^I(t) \to v_0$ will collapse a triangle subgraph that contains one interior vertex v_1^I.

Lemma 4.4.1 *Consider a boundary vertex v_0 of (Ω, T). Then there is an interior vertex $v_1^I \in N(v_0)$ such that it is not a boundary vertex of a triangle subgraph which contains an interior vertex of (Ω, T) within the region bounded by the boundary of this subgraph. In other words, there is no $v_2 \in V$ for which the triangle subgraph $\Delta := \Delta v_0 v_1^I v_2$ both exists and has any $v_i^I \in V_I$ within its interior.*

Proof. By way of contradiction, suppose that for every interior vertex $v_i \in N(v_0)$ there is such a $v_{i,2} \in N(v_i^I) \cap N(v_0)$ for which the triangle subgraph $\Delta_i := \Delta v_0 v_i^I v_{i,2}$ exists and there are interior vertices of (Ω, T) contained within the region bounded by this triangle. Consider any such, which we can label Δ_1. Because Δ_1 is triangulated by definition, there is a vertex

$v_2 \in V$ interior to Δ_1 such that it is the boundary vertex of a triangle $\Delta_2 \subset \Delta_1$. Continuing in this manner with the existence of a vertex $v_3 \in V$ within Δ_2 and a corresponding triangle $\Delta_3 \subset \Delta_2$ with v_3 as one of its boundary vertices, we eventually achieve a contradiction as the number of interior vertices of (Ω, T) is finite. ■

As such, for any $v_1^B \in V_B$ there is a $v_1^I \in N(v_1^B) \cap V_I$ for which deforming w_{11}^{IB} does not collapse a subcomplex in the manner described above.

When this v_1^I collapses to v_1^B via the process introduced in the previous section, it is the case that two interior edges adjacent to v_1^I degenerate to either

- two boundary edges,
- one interior edge and one boundary edge, or
- two interior edges,

as depicted in Figure 7 below. Consequently, $\lim_{t\to\infty} \tau(t) \notin X(\Omega, T)$ as one might expect. Regardless, we still want to examine the combinatorics of this limiting behavior, and to do so we must study the edge weight of these aforementioned edges adjacent to v_1^I.

In the first case where they both collapse to two boundary edges, then there is no concern as edge weights are only assigned to interior edges. As such, we need only consider the two cases where either one or both of these strictly interior edges collapse to another interior edge. In the following Proposition, we consider the case where these two aforementioned interior edges adjacent to v_1^I collapse to one interior edge and one boundary edge. (The other case can be resolved in a similar manner.)

Proposition 4.4.1 *Let Δ denote the triangle subgraph formed by v_0, v_1^I, and v_2^I, if any such triangle exists. (Recall that by construction, this subgraph has empty interior.) Then as $t \to \infty$, the interior edge $e_{12}^{II} = e_{21}^{II}$ is identified to e_{20}^{IB} and the corresponding edge weight of this new interior-to-boundary edge is precisely $w'_{20} := w_{21}^{II} + w_{20}^{IB}$.*

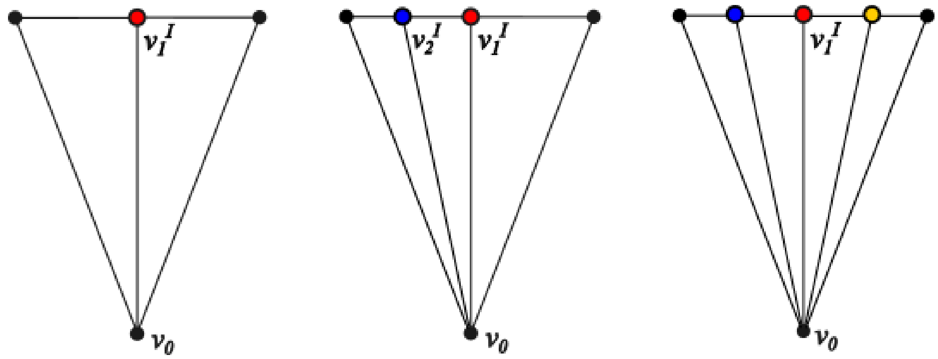

Figure 7: Letting $v_1^I(t) \to v_0$ collapses two interior edges adjacent to v_1^I to two other edges.

Proof. Recall that we have the augmented Tutte system of equations

$$0 = \sum_{v_i \in N(v_k^I)} w_{ki}(t)(v_k^I(t) - v_i(t)),$$

for every $1 \leq k \leq N_I$. (Of course, if v_i is a boundary vertex then $v_i(t) = v_i$.)

If we take the limit of the above system as $t \to \infty$, then for every $k \neq 1, 2$ we have as our limiting value

$$0 = \sum_{v_i \in N(v_k^I)} w_{ki}(v_{k,\infty}^I - v_{i,\infty}), \tag{4.4.1}$$

where $v_{i,\infty} = \lim_{t\to\infty} v_i(t)$. From Corollary 4.3.1, the equation for $k = 1$ vanishes as $v_{1,\infty}^I = v_0$ is no longer interior in Ω.

Finally, for $k = 2$ we have

$$\begin{aligned} 0 &= w_{20}(v_{2,\infty}^I - v_0) + w_{21}(v_{2,\infty}^I - v_{1,\infty}) + \sum_{v_i \in N(v_2^I)\setminus\{v_0, v_1\}} w_{2i}(v_{k,\infty}^I - v_i) \\ &= w'_{20}(v_{2,\infty}^I - v_0) + \sum_{v_0 \neq v_i \in N(v_{2,\infty}^I)} w_{2i}(v_{2,\infty}^I - v_i), \end{aligned} \tag{4.4.2}$$

as required. ■

Consequently, the $N_I - 2$ equations (4.4.1) and the one equation (4.4.2) comprises a Tutte system of equations with $N_I - 1$ equations in $N_I - 1$ unknowns $v_{2,\infty}^I, \ldots, v_{N_I,\infty}^I$, because the new set of coefficients $w' = \{w'_{ki}\}$ in this system are

- For $k = 2$: If $i = 0$ then $w'_{20} := w_{20} + w_{21}$ and $w'_{2i} := w_{2i}$ otherwise if $v_i \sim v_{2,\infty}^I$;

- For $2 < k \leq N_I : w'_{ki} := w_{ki}$ if $v_i \sim v^I_{k,\infty}$.

Because every entry of w' is non-negative, then it is a weight of (Ω, T') so that the above system has a unique solution $\tau_\infty = (v^I_{2,\infty}, \ldots, v^I_{N_I,\infty}) \in X(\Omega, T')$ by Tutte's Embedding Theorem, where T' is specified below.

To be more precise in our definition of the triangulation T' of Ω that is obtained from T by identifying v^I_1 to v_0 in the above deformation – and to show that it is indeed a triangulation of Ω – observe from Figure 7 that this process results in the loss of

- three interior edges of T, one of which is e^{IB}_{10},
- two faces of T, and
- one interior vertex of T.

As such, the Euler characteristic of the resulting connected planar graph (Ω, T') is then

$$(V - 1) - (E - 3) + (F - 2) = V - E + F,$$

which is exactly the Euler characteristic of (Ω, T). In other words, the obtained combinatorics T' of Ω is still a triangulation of Ω.

This yields our main result of this Chapter:

Theorem 4.4.1 *Define the function* $f_\infty : X(\Omega, T) \to X(\Omega, T')$ *by*

$$f_\infty(\tau) = \tau_\infty := \lim_{t \to \infty} \tau(t).$$

Then f_∞ *is both well-defined and continuous.*

4.5 Motion of the Interior Vertices

One interesting characteristic of $\tau(t)$ pertains to the motion of its interior vertices as $t \to \infty$.

Proposition 4.5.1 *As a function of t, the direction of each $v_k^I(t) \in V_I$ is parallel to that of $v_1^I(t)$. Moreover, this latter vertex moves at the fastest speed relative to every other interior vertex.*

Proof. First, we compute the matrix derivative

$$\begin{aligned}\frac{d}{dt}x^I(t) &= \frac{d}{dt}S^{-1}(t)W(t)x^B \\ &= \left(\left(\frac{d}{dt}S^{-1}(t)\right)W(t) + S^{-1}(t)\frac{d}{dt}W(t)\right)x^B \\ &= S^{-1}(t)\left(-\begin{bmatrix}1 & \\ & 0\end{bmatrix}S^{-1}(t)W(t)x^B + \begin{bmatrix}1 & \\ & 0\end{bmatrix}x^B\right)\end{aligned}$$

which, using the first equation in (4.2.2), reduces to

$$\begin{aligned}\frac{d}{dt}x^I(t) &= S^{-1}(t)\left(-\begin{bmatrix}1 & \\ & 0\end{bmatrix}x^I(t) + \begin{bmatrix}1 & \\ & 0\end{bmatrix}x^B\right) \\ &= \begin{bmatrix}S^{-1}_{11}(t) \\ \vdots \\ S^{-1}_{N_I 1}(t)\end{bmatrix}(x_1^B - x_1^I(t)).\end{aligned} \tag{4.5.1}$$

A similar calculation holds for $\frac{d}{dt}y^I(t)$; from these two derivatives, we conclude that the motion of every $v_k^I(t)$ is always parallel to that of $v_1^I(t)$.

To prove the second claim, we use the following Lemma due to Shivakumar:

Lemma 4.5.1 (Shivakumar [13]) *For a weakly-chained diagonally dominant M-matrix A, $A^{-1}_{ij} \leq A^{-1}_{jj}$.*

An application of this Lemma to S yields $S_{11}^{-1}(t) \geq S_{j1}^{-1}(t)$ for every $2 \leq j \leq N_I$. Combining this inequality with (4.5.1) implies $v_1^I(t)$ has the greatest speed. ■

We conclude this Chapter by establishing the following Proposition, which shows the path of each interior vertex $v_i^I(t)$ as we perturb t:

Proposition 4.5.2 *Every $v_i^I(t)$, for $1 \leq i \leq N_I$, travels in a straight-line motion.*

Proof. Observe that, from Equation (4.5.1), the interior vertex $v_i^I(t)$ has initial direction $v_1^B - v_1^I(0)$. For the sake of contradiction, suppose that this vertex does not move in a straight-line motion as a function of t and let t_0 denote the first time value for which $v_i^I(t)$ does not point in the direction $v_1^B - v_1^I(0)$. However, this contradicts the fact that $v_i^I(t)$ has direction $v_1^B - v_1^I(t_0)$ at $t = t_0$. ■

CHAPTER V

STAR-SHAPED POLYGONS

In this Chapter, we briefly discuss a possible approach toward showing that $X(\Omega, T)$ is contractible, where (Ω, T) is a triangulated strictly star-shaped polygon with one reflex vertex $v^B_{N_B} = v_r = (x_r, y_r) \in \mathbb{R}^2$ and T does not contain any dividing edges.

5.1 The Weight Space

The primary reason why we avoid directly manipulating the coordinates of the interior vertices of elements in $X(\Omega, T)$ is that it is difficult to systematically perturb them while avoiding collisions with edges or other vertices – moreso if N_I is very large. Instead, we wish to work with the weight space $W(\widetilde{\Omega}, \widetilde{T})$ of a triangulated convex polygon $(\widetilde{\Omega}, \widetilde{T})$ which is related to (Ω, T) in the manner depicted below, the two upsides being that $W(\widetilde{\Omega}, \widetilde{T})$ is convex and, if we deform a $\tau = \Psi(w) \in X(\widetilde{\Omega}, \widetilde{T})$, for some weight $w \in W(\widetilde{\Omega}, \widetilde{T})$, by varying the edge weight of a certain interior-to-boundary edge in w in the manner constructed in Chapter IV, then we can continuously map τ to a similar $\tau' \in X(\widetilde{\Omega}, \widetilde{T})$ without any degeneracies.

5.2 Convex Hull of a Star-Shaped Polygon

To that end, we define $(\widetilde{\Omega}, \widetilde{T})$ as follows:

- Let $\widetilde{\Omega}$ be the convex polygon obtained by taking the convex hull of Ω – that is, we form a boundary edge connecting the only two boundary vertices $v^B_{N_B-1}$ and $v^B_{N_B-2}$ adjacent to v_r;
- Because v_r is in the interior of $\widetilde{\Omega}$, form the triangulation $\widetilde{T}$ of $\widetilde{\Omega}$ by now treating v_r as

an interior vertex of T, which we denote by $\tilde{v}_r = v^I_{N_I+1}$, and the two boundary edges $e^{BB}_{N_B-1,N_B}$, $e^{BB}_{N_B-2,N_B}$ of (Ω, T) as now interior-to-boundary edges of T, which is denoted by $e^{IB}_{N_I+1,N_B-1}$ and $e^{IB}_{N_I+1,N_B-2}$, respectively.

This yields the following relation:

Lemma 5.2.1 *There exists an embedding* $X(\Omega, T) \hookrightarrow X(\widetilde{\Omega}, \widetilde{T})$.

Proof. This is given by $(v^I_1, v^I_2, \ldots, v^I_{N_I}) \longmapsto (v^I_1, v^I_2, \ldots, v^I_{N_I}, v_r)$. ■

Note that the ambient space $X(\widetilde{\Omega}, \widetilde{T})$ is contractible by Corollary 1.3.1. As such, we conjecture that

Conjecture 5.2.1 *There is a deformation retract*

$$D : X(\widetilde{\Omega}, \widetilde{T}) \times I \to X(\widetilde{\Omega}, \widetilde{T})$$

from $X(\widetilde{\Omega}, \widetilde{T})$ *to* $X(\Omega, T)$.

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